Simply and Multiply Connected Domains

Ref: Complex Variables by James Ward Brown and Ruel V. Churchil

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46. Simply and multiply connected domains

Definition: A **simply connected domain** D is a domain such that every simple closed contour within it encloses only points of D.

Example: The set of points interior to a simple closed contour is a simply connected domain. The annular domain between two concentric circles is not simply connected.

Definition: A domain that is not simply connected is said to be multiply connected.

Theorem1: (Extension of Cauchy-Goursat theorem)

If a function f is analytic throughout a simply connected domain D, then

$\int_C f(z)dz = 0$ --- (1) for every closed contour C lying in D.

Proof:

Case1 C is a simple closed contour

If C lies in D, the function f is analytic at each point interior to and on C; and the Cauchy-Goursat theorem ensures that $\int_C f(z)dz = 0$.

Case2 C is a closed contour that intersects itself finite number of times

In this case it consists of a finite number of simple closed contours.

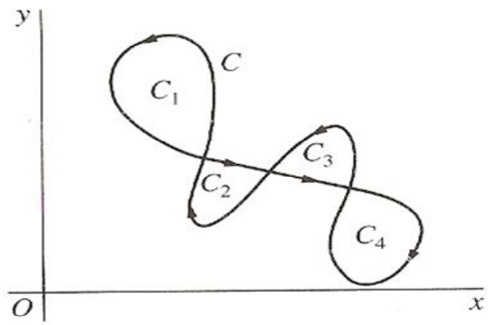

FIGURE 57

In the figure, the simple closed contours C_k (k=1,2,3,4) make up C. The value of the integral around each C_k is zero by Cauchy-Goursat theorem. Hence

$$\int_C f(z)dz = \sum_{k=1}^{4} \int_{C_k} f(z)dz = 0 . \quad \blacksquare$$

Corollary1: A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.

Proof:

Given: f is analytic throughout a simply connected domain.

$$\Rightarrow \int_C f(z)dz = 0$$ for each closed contour C in that domain.

$\Rightarrow$ f has an antiderivative in that domain(since f is analytic $\Rightarrow$ f is continuous). ■

Remark: The finite plane is simply connected. So by Corollary1, entire functions always possess antiderivatives.

The Cauchy-Goursat theorem can also be extended in a way that involves integrals along the boundary of a multiply connected domain. The following theorem is such an extension.

Theorem2: Suppose that (i) C is a simple closed contour, described in the counter clockwise direction. (ii) C_k (k=1,2,…,n) are simple closed contours interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no points in common.

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of all points inside C and exterior to each C_k, then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0 \quad \text{--- (2).}$$

Proof:

Note that in equation (2), the direction of each path of integration is such that the multiply connected domain lies to the left of that path.

To prove the theorem, we introduce a polygonal path L_1, consisting of a finite number of line segments joined end to end, to connect the outer contour C to the inner contour C_1. We introduce another polygonal path L_2 which connects C_1 and C_2 ; and we continue in this manner, with L_{n+1} connecting C_n to C.

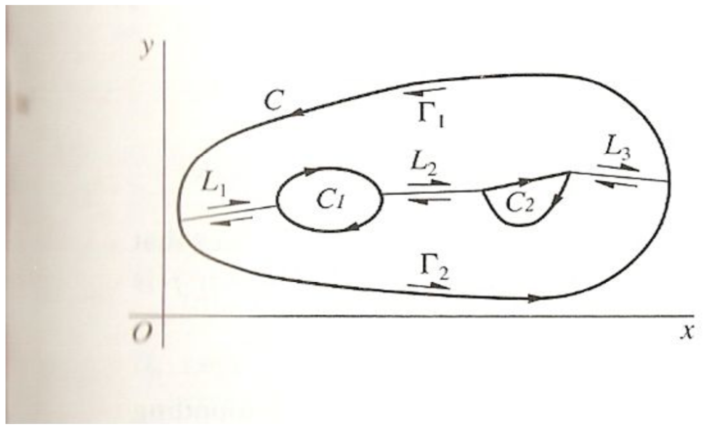

FIGURE 58

As indicated by the single-barbed arrows in Figure 58, two simple closed contours Γ_1 *and* Γ_2 can be formed, each consisting of polygonal paths L_k or $-L_k$ and pieces of C and C_k and each described in such a direction that the points enclosed by them lie to the left.

By Cauchy-Goursat theorem

$$\int_{\Gamma_1} f(z)dz = 0 \text{ and } \int_{\Gamma_2} f(z)dz = 0$$

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0$$

$$\Rightarrow \int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$ (Since the integrals in opposite directions along each

path L_k cancel, only the integrals along C and C_k remain). ■

Corollary2: (Principle of deformation of paths)

Let C_1 and C_2 denote positively oriented simple closed contours, where C_2 is interior to C_1. If a function f is analytic in the closed region consisting of those contours and all points between them, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ --- (3)

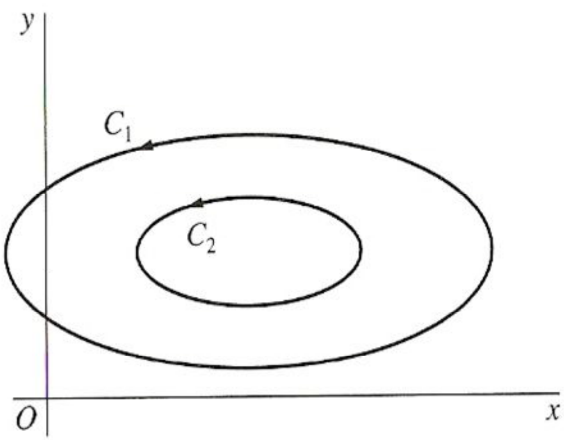

FIGURE 59

Proof: We know that "Suppose that (i) C is a simple closed contour, described in the counter clockwise direction. (ii) C_k (k=1,2,…,n) are simple closed contours interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no points in common.

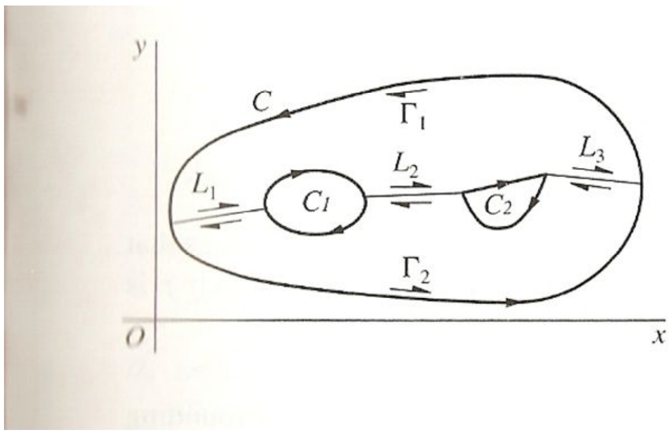

FIGURE 58

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of all points inside C and exterior to each C_k, then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0 \text{ “}$$

$$\Rightarrow \int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0$$

$$\Rightarrow \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

$$\Rightarrow \int_{C_1} f(z)dz = \int_{C_2} f(z)dz \,. \qquad \blacksquare$$

Remark: Corllary2 is known as **the principle of deformation of paths** since it tells us that if C_1 is continuously deformed into C_2, always passing through points at which f is analytic, then the value of the integral of f over C_1 never changes.

Example: When C is any positively oriented simple closed contour surrounding the origin, show that $\int_C \frac{dz}{z} = 2\pi i.$

Solution: If C_0 is $|z| = \rho$ then $z = \rho e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

Now $\int_{C_0} \frac{dz}{z} = \int_0^{2\pi} \frac{\rho i\, e^{i\theta} d\theta}{\rho e^{i\theta}} = i \int_0^{2\pi} d\theta = 2\pi i.$

If ρ is small such that C_0 lies entirely inside C, by the principle of deformation of paths,

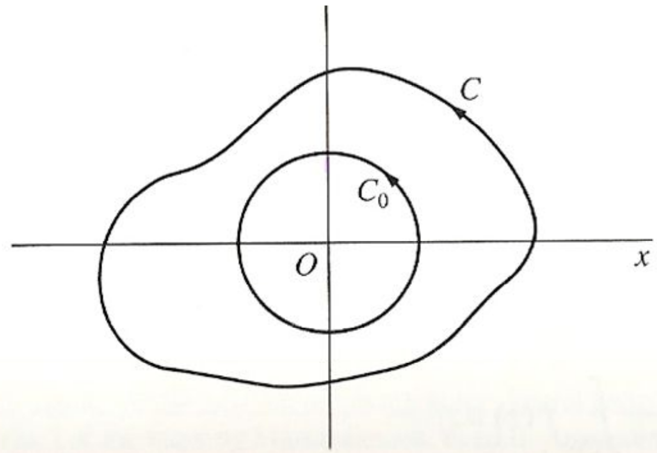

FIGURE 60

$$\int_C \frac{dz}{z} = \int_{C_0} f(z)dz = 2\pi i.$$

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